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p-Adic Stochastics and Dirac Quantization with Negative Probabilities

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A new mathematical apparatus, a p-adic theory of probability, is applied to realize the hypothetical world based on negative probability distributions created by Dirac for the relativistic quantization of photons. Within the p-adic theory of probability, negative probability distributions are well defined (in the language of limits of relative frequencies, but with respect to a p-adic metric). We propose that the negative Gibbs distributions arising in relativistic quantization are described by p-adic stochastics.

1. INTRODUCTION

Negative probabilities of certain states arose in a natural way in the relativistic quantization of photons (Dirac, 1942). Dirac proposed to consider a hypothetical world with negative probabilities as a mathematical idealization to resolve the problem of negative energy. This mathematical construction arose so naturally that he was sure that it must have some probability meaning. But it would be impossible to physically justify a quantum state with negative "probability" of realization because this "probability" has no probability sense. A probability according to the Kolmogorov (1933) axiomatics is a positive-definite measure. Dirac (1942) did not consider the problem of negative probabilities in a rigorous way. He confined himself to noting that probably it would be similar to a negative sum of money, for example.

For a long time the hypothetical Dirac world with negative probability distributions was considered as an interesting construction which can be useful in some cases but has no direct sense. In this paper we propose another point of view on negative probability distributions in the relativistic

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quantization of photons. Our considerations are based on p-adic numbers [on these numbers see, for example, Mahler (1973) or Schikhof (1984)], p-adic-valued quantum mechanics (Khrennikov, 1991, 1994a,b), and the p-adic-valued theory of probability (see the previous references and Khrennikov, 1989, 1990).

p-Adic physical models attempt to describe reality with the aid of a number field Q_p which has many properties very different from the real or the complex case. The first results in this direction were the papers of Beltrametti and Cassinelli (1972), Vladimirov and Volovich (1984), Volovich (1987), Freund and Witten (1987), and Frampton and Okada (1988); see also Vladimirov *et al.* (1994). The main activity was connected with the theory of p-adic strings. The complex-valued amplitudes and wave functions (of string and later quantum mechanics and field theory) on p-adic space-time was the main object of these investigations. I consider another model of p-adic physics and study quantum mechanics with p-valued wave functions [p-adic-valued string amplitudes were also constructed in Volovich (1987)].

The main problem of p-adic quantum mechanics is the probability interpretation of these wave functions. A new mathematical theory, a p-adic-valued theory of probability, was proposed to resolve this problem. As usual we consider a probability as a limit of relative frequencies ν_n, but with respect to another metric on the field of rational numbers Q.

We base our game on the following evident fact. The only physical numbers are rational numbers. We can get in any experiment only finite fractions and not more general real, complex, (or p-adic) numbers. Then we can study these rational data with the aid of different mathematical methods. p-Adics helps us to find some additional information about these rational numbers which we cannot find on the basis of real numbers. In particular, there exist random sequences where ν_n oscillates between 0 and 1 with respect to the usual real metrics, but stabilize with respect to one of the p-adic metrics; Khrennikov (1994a) gives also the results of computer simulations. From the usual point of view such sequences are chaotic; it would be impossible to define real probabilities. But there is a well-defined probability distribution from the p-adic point of view.

p-Adic probability has many of the properties of standard probability: additivity, conditional probabilities, independent events, etc. But one property is very different. In the usual case all probabilities belong to the segment $[0, 1]$ of R. This fact is evident. Since relative frequencies obey $0 \leq \nu_n \leq 1$, their limit is also between 0 and 1. But we can represent every rational number as the limit of ν_n with respect to the p-adic metric (Khrennikov, 1994a). And in particular, every negative rational number can be presented as the limit of relative frequencies. This fact is the cornerstone of our further considerations. The negative probability distributions are ordinary objects in the p-adic theory of probability. They have the same statistical connection with relative frequencies and negative probabilities, which is impossible in the usual theory of probability.

2. p-ADIC NUMBERS

The field of real numbers R is constructed as a completion of the field of rational numbers Q with respect to the metric $\rho(x, y) = |x - y|$, where $|\cdot|$ is the usual absolute value norm. Fields of p-adic numbers Q_p are constructed in the same way. There is an infinite sequence of p-adic number fields, a field for every prime number $p = 2, 3, 5, \ldots$. A p-adic norm $|\cdot|_p$ is defined in the following way.

First, we define it for natural numbers. Every natural number n can be represented as the product of prime numbers: $n = 2^{r_2}3^{r_3} \cdots p^{r_p} \cdots$. Then $|n|_p = p^{-r_p}$, by the definition $|0|_p = 0$, $|-n|_p = |n|_p$, and $|n/m|_p = |n|_p/|m|_p$. The completion of Q with respect to the metric $\rho_p(x, y) = |x - y|_p$ is a locally compact field Q_p.

It is an intrinsic fact (Ostrovsky theorem) of the theory of numbers (see, e.g., Mahler, 1973; Schikhof, 1984; Borevich and Schafarevich, 1966) that the only norms on Q are the real one $|\cdot|$ or a p-adic one.

As for real numbers, there exists a canonical expansion of p-adic numbers,

$$x = a_{-n}/p^n + \cdots a_0 + \cdots + a_k p^k + \cdots \tag{1}$$

where $a_j = 0, 1, \ldots, p - 1$ are digits of the p-adic expansion. In a real case such an expansion is infinite in the direction of negative degrees and p-adic in the positive direction, and in both cases such an expansion is unique. The expansion (1) is the basis of the p-adic statistical simulation. In a sense the negative probabilities can arise in the same way as $-1 = 1 + 2 + 4 + \ldots$, and this series converges in Q_2. The following result will be very useful ("a dream of a bad student"): a series $\Sigma\, w_n$, $w_n \in Q_p$, converges **iff** $|w_n|_p \to 0$, $n \to \infty$.

Using the definition of the p-adic valuation, we get $|n| \leq 1$ for every natural number n. Thus the sequence $|n!|_p$ is decreasing. Moreover, we have

$$p^{(n-1)/(1-p)} \leq \left|n!\right|_p \leq npp^{n/(1-p)} \tag{2}$$

Let us suppose that the quadratic equation $x^2 - \tau = 0$, $\tau \in Q_p$, has no solution in the field Q_p. We use the symbol Z_τ for the quadratic extension $Q_p(\sqrt{\tau})$ of the field Q_p. The elements of Z_τ are represented as $z = x + \sqrt{\tau}y$, $x, y \in Q_p$, and the conjugation operation is $\bar{z} = x - \sqrt{\tau}y$; the valuation on Z_τ is also denoted by $|\cdot|_p$, $|z|_p = (||z|^2|_p)^{1/2}$. Introduce also an analog of the Euclidean square length $|z|^2 = z\bar{z} = x^2 - \tau y^2$. This square length assumes its values in the field Q_p (but $|\cdot|_p$ assumes its values in the field of real numbers).

We shall be interested in square roots of $x = -1$ in Q_p. This square root exists in Q_p if $p = 1 \pmod 4$. Thus, if $p \neq 1 \pmod 4$, then this square root does not exist in Q_p and we can use the quadratic extension $Z_i = Q_p(i)$ as the field of the complex p-adic numbers. This choice will be convenient for realizing the Dirac quantum states.

3. ANALYSIS OF THE FOUNDATIONS OF THE THEORY OF PROBABILITY AND A p-ADIC THEORY OF PROBABILITY

The basis of the modern theory of probability is a measure-theoretic axiomatics (Kolmogorov, 1933). A probability is a normalized measure μ assuming its values in the segment $[0, 1] \subset R$. But why? To formulate his axioms, Kolmogorov used the properties of frequency probability (von Mises, 1919, 1964).

A basic object of von Mises' theory is a *collective*. Let $\mathcal{S}$ be a random experiment and $T = \{\alpha_1, \ldots, \alpha_m\}$ be a set of all possible realizations of $\mathcal{S}$ (in the simplest case $T = \{0, 1\}$). An infinite sequence of realizations of $\mathcal{S}$

$$x = (x_1, \ldots, x_n, \ldots), \qquad x_i \in T \tag{3}$$

is said to be a collective if for every label $\alpha \in T$ there exists a limit of relative frequencies $\nu_n(\alpha) = k(\alpha)/n$: $\mathbf{P}(\alpha) = \lim_{n\to\infty} \nu_n(\alpha)$. This limit is said to be a probability of α. In particular, why does a probability belong to the segment $[0, 1]$ of the real line in the Kolmogorov axiomatics? No problem; $0 \leq \nu_n(\alpha) \leq 1$ and this is why $0 \leq \mathbf{P} \leq 1$.

Now a sequence (3) is said to be a p-adic collective (Khrennikov, 1989, 1990, 1994a) if a limit of relative frequencies exists with respect to the p-adic metric for every label $\alpha \in A$. This limit $\mathbf{P}_p(\alpha) = \lim_{n\to\infty} \nu_n(\alpha)$ is called a p-adic probability. p-Adic collectives are considered in Khrennikov (1994a) from the theoretical and computer simulation points of view. In a series of computer statistical experiments, relative frequencies oscillated between 0 and 1 with respect to a real metric, but stabilize very quickly with respect to one of the p-adic metrics (p plays the role of a parameter of the model).

A measure-theoretic definition of a p-adic probability is generated by the frequency definition. It is a Q_p-valued normalized measure [on Q_p-valued measures see Schikhof (1984)]. An example of a p-adic-valued distribution with negative probabilities of some events is proposed in the Appendix.

4. A *p*-ADIC HILBERT SPACE AND *p*-ADIC-VALUED QUANTIZATION

The definition of the *p*-adic Hilbert space (Khrennikov, 1991, 1993, 1994a,b) is based on the coordinate representation (an analog of l_2). For the sequence $\lambda = (\lambda_n) \in Q_p^\infty$, $\lambda_n \neq 0$, we set

$$\mathcal{H}_\lambda = \{f = (f_n)\text{: the series } \sum f_n^2 \lambda_n \text{ converges}\}$$

We have

$$\mathcal{H}_\lambda = \{f = (f_n)\text{: } \lim_{n\to\infty} |f_n|_p (|\lambda_n|_p)^{1/2} = 0\}$$

In the space $\mathcal{H}_\lambda$ we introduce a norm $\|f\|_\lambda = \max_n |f_n|_p (|\lambda_n|_p)^{1/2}$. The space $\mathcal{H}_\lambda$ is a non-Archimedean Banach space. On the space $\mathcal{H}_\lambda$ we introduce an inner product $(\cdot,\cdot)$ consistent with the length $|f|_\lambda^2 = \Sigma f_n^2 \lambda_n$, setting $(f, g)_\lambda = \Sigma f_n g_n \lambda_n$. The inner product $(\cdot,\cdot)$: $\mathcal{H}_\lambda \times \mathcal{H}_\lambda \to Q_p$ is continuous and we have the *Cauchy–Buniakovski inequality*:

$$|(f, g)_\lambda|_p \leq \|f\| \cdot \|g\| \tag{4}$$

Definition 1. The triplet $(\mathcal{H}_\lambda, (\cdot,\cdot)_\lambda, \|\cdot\|_\lambda)$ is called a coordinate Hilbert space.

An inner product on the Q_p-linear space E is an arbitrary nondegenerate symmetric bilinear form $(\cdot,\cdot)$: $E \times E \to Q_p$. It is evidently impossible to introduce an analog of the positive definiteness of a bilinear form. For instance, for the field of *p*-adic numbers any element $\gamma \in Q_p$ can be represented as $\gamma = (x, x)_\lambda$, $x \in \mathcal{H}_\lambda$ [this is a simple consequence of properties of bilinear forms over Q_p; see, for example, Borevich and Schafarevich (1966)]. Weight coefficients λ will be rational numbers in all applications. There is no difference between inner products $(\cdot,\cdot)_\lambda$ for λ consisting of positive rational numbers and mixed positive-negative. For example, if all $\lambda_j = 1$, then there exist nonzero vectors $x \in \mathcal{H}_\lambda$ such that $(x, x)_\lambda = 0$. The same holds for the normalization of basis vectors $e^j = (e_i^j) = (\delta_i^j)$. These vectors are orthogonal, but $|e^j|^2 = (e^j, e^j) = \lambda_j$. The standard normalization is impossible also for positive rational coefficients, since a square root $\sqrt{\lambda_j}$ may not exist in Q_p. From the *p*-adic point of view the weight sequence $\lambda_j = n!$ has the same problems of normalization as $\lambda_j = (-1)^n n!$. Thus, the weight sequence plays an important role in our case (see further the investigation of the Dirac relativistic quantization of photons).

The triplets $(E_j, (\cdot,\cdot)_j, \|\cdot\|_j)$, $j = 1, 2$, where E_j are non-Archimedean Banach spaces, $\|\cdot\|_j$ are norms, and $(\cdot,\cdot)_j$ are inner products satisfying (4), are isomorphic if the spaces E_1 and E_2 are algebraically isomorphic and the algebraic isomorphism $I: E_1 \to E_2$ is isometric and unitary, i.e., $\|Ix\|_2 = \|x\|_1$, $(Ix, Iy)_2 = (x, y)_1$.

Definition 2. The triplet $(E, (\cdot,\cdot), \|\cdot\|)$ is a p-adic Hilbert space if it is isomorphic to the coordinate Hilbert space $(\mathcal{H}_\lambda, (\cdot,\cdot)_\lambda, \|\cdot\|_\lambda)$ for a certain λ.

The isomorphic relation divides the class of Hilbert spaces into equivalence classes. We shall define the equivalence class of Hilbert spaces by some coordinate representative $\mathcal{H}_\lambda$.

Example 1. Let $\lambda = (1)$ and $\mu = (2^n)$. The spaces $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ belong to the same class of equivalence for the field Q_p, $p \neq 2$, and to different classes for the field Q_2.

But it is a hard, unresolved mathematical problem to classify p-adic Hilbert spaces.

Hilbert spaces over quadratic extensions $Z_\tau = Q_p(\sqrt{\tau})$ can be introduced by analogy. For the sequence $\lambda = (\lambda_n) \in Q_p^\infty$, $\lambda_n \neq 0$, we set

$$\mathcal{H}_\lambda = \{f = (f_n) \in Z_\tau^\infty: \text{the series } \sum |f_n|^2\lambda_n \text{ converges in the field } Q_p\}$$

$$= \{f = (f_n): \lim_{n\to\infty} |f_n|_p(|\lambda_n|_p)^{1/2} = 0\}$$

$$\|f\|_\lambda = \max_n |f_n|_p(|\lambda_n|_p)^{1/2}$$

$$(f, g) = \sum f_n\bar{g}_n\lambda_n; \qquad |f|_\lambda^2 = (f, f)_\lambda = \sum |f_n|^2\lambda_n \in Q_p$$

The triplet $(\mathcal{H}_\lambda, (\cdot,\cdot)_\lambda, \|\cdot\|_\lambda)$ is a non-Archimedean complex coordinate Hilbert space. The non-Archimedean complex coordinate Hilbert space $(E, (\cdot,\cdot), \|\cdot\|)$ is defined as an isomorphic image of a coordinate Hilbert space.

Note that the normalization problem is present also in the case of complex p-adic Hilbert spaces. For example, it is also impossible to normalize weight sequence $\lambda_j = n!$.

Now we can consider p-adic complex Hilbert spaces as the state spaces of p-adic-valued quantum mechanics; observables are symmetry operators. The connection with ordinary quantum mechanics is based on the rational numbers. As all observables in real experiments have rational values, we can realize these observables in both ordinary quantum mechanics and in the p-adic one. What is the difference between the two formalisms? We investigate new asymptotic properties using p-adic numbers. The p-adic Hilbert space consists of new quantum states which were impossible to realize in ordinary quantum mechanics.

The statistical interpretation is based on the p-adic theory of probability. Let $f \in \mathcal{H}_\lambda$, $|f| = (f, f)_\lambda = 1$. In a long series of experiments relative frequencies $\nu_N(j) = n(j)/N$ of realizations of the states e^j approach the p-adic probabilities $\mathbf{P}_p = \lambda_j |f_j|^2$.

We can consider a quantum mechanical model based on p-adic Hilbert space as the extension of the superposition principle. We use new linear combinations of eigenstates which do not belong to the ordinary Hilbert space (it is possible to consider only rational coordinates).

5. THE DIRAC FORMALISM AND ITS p-ADIC PROBABILITY INTERPRETATION

To delete the divergences, Dirac proposed to consider the representation including positive and negative energies. Then to resolve the problem of negative energies, he proposed to consider operators of emission of photons with negative energy as absorption operators of photons with positive energy. But this picture contains negative probabilities of absorption of any odd number of photons.

Let $A^1(x)$ be operators of the quantum electrodynamics of Heisenberg and Pauli referring to the emission and absorption of photons into positive-energy states:

$$A^1(x) = \iiint (R_k e^{(k,x)} + \bar{R}_k e^{-(k,x)}) k_0^{-1} dk_1 dk_2 dk_3 \tag{5}$$

where $k_0 = +(k_1^2 + k_2^2 + k_3^2)^{1/2}$ and R_k is the emission operator and $\bar{R}_k$ the absorption operator. In the same way it is possible to introduce the operators $A^2(x)$ referring to the negative energy; there is a representation similar to (5) but with $k_0 = -(k_1^2 + k_2^2 + k_3^2)^{1/2}$. Dirac considered operators $A^3 = (1/\sqrt{2})(A^1 + A^2)$ which are expanded with respect to operators R_k and $\bar{R}_k$ corresponding to positive and negative energies.

The idea was to resolve all divergence problems in the symmetric $A^3(x)$ representation. Then there is the possibility to get information about the $A^1(x)$ representation. But it would be impossible to apply the linear transformation between the $A^3(x)$ and $A^1(x)$ representations to the wave function of the $A^3(x)$ representation. There would arise the same divergences. But it is possible to do this with the initial Gibbs ensemble of the $A^3(x)$ representation.

It is convenient to consider with $A^3(x)$ the additional fields $B^3(x) = (1/\sqrt{2})[A^1(x) - A^2(x)]$, which commute with $A^3(x)$, so they are redundant variables. Now let us take B equal to the initial value of A^3. Then for the initial wave function ψ,

$$(B^3(x) - A^3(x))\psi = 0$$

or $\bar{R}_k\psi = 0$ with k_0 either positive or negative. Thus any absorption operator applied to the initial wave function gives the result zero, which means that the corresponding state is one with no photons present.

The following natural interpretation of the wave function at some later time now appears. That part corresponding to m photons of positive energy and n photons of negative energy can be interpreted as corresponding to m photons having been emitted and n photons having been absorbed. This confirms to the laws of conservation of energy and momentum.

Then Dirac considered the momentum representation of $A^3(x)$ and $B^3(x)$ operators. Let k be a momentum-energy vector, $k^2 = 0$, and $\xi_{k\mu}$, $\xi^*_{k\mu}$ be operators of emission and absorption. We have $k_0 = \pm(k_1^2 + k_2^2 + k_3^2)^{1/2}$. Then set $\zeta_{k\mu} = \xi_{-k\mu}$ for $k_0 > 0$ and consider the wave function ψ as $\psi = \psi(\xi, \zeta)$, $k_0 > 0$. The following commutation relations hold: $[\xi^*, \xi] = c$ and $[\zeta^*, \zeta] = -c$, $c > 0$.

The variable ξ corresponds to the emission of photons of positive energy, $k_0 > 0$; and ζ corresponds to the absorption of photons of positive energy, $k_0 > 0$. Let us denote the space of states $\psi(\xi, \zeta)$ by the symbol $\mathcal{H}$. The Dirac inner product in $\mathcal{H}$ has the form

$$(f, g) = \sum_{m,n=0}^{\infty} f_{mn} \bar{g}_{nm} m!\, c^m n!\, (-c)^n \tag{6}$$

for the functions

$$f(\xi, \zeta) = \sum_{mn} f_{mn} \xi^m \zeta^n, \qquad g(\xi, \zeta) = \sum_{mn} g_{mn} \xi^m \zeta^n \tag{7}$$

Now for the wave function $\psi(\xi, \zeta)$, normalized by $|\psi|^2 = (\psi, \psi) = 1$, the probability of there having been m photons emitted into momentum and energy state k (corresponding to ξ) and n photons absorbed from this state is

$$\mathbf{P}(m, n) = |\psi_{mn}|^2 c^m m!\, (-c)^n n! \tag{8}$$

This gives a negative probability for an odd number of photons to have been absorbed. But this statistical interpretation has no sense within the ordinary theory of probability. This is why Dirac considered Gibbs ensembles of this type as ideal mathematical objects. But at the same time he was surprised that this ideal Gibbs ensemble is very similar to usual physical Gibbs ensembles.

We use p-adic quantum theory to realize these Gibbs ensembles as real Gibbs ensembles, which can exist in nature, but describe them with the aid of another type of stochastics, p-adic [on the results of computer statistical simulation of p-adic distributions see Khrennikov (1994a)].

Now let us realize the Dirac state space $\mathcal{H}$ as the p-adic Hilbert space $\mathcal{H}_\lambda$, where $\lambda = (\lambda_{nm})$ is a two-index weight sequence and $\lambda_{nm} = c^m m!\, (-c)^n n!$ and we consider the case of a rational number $c > 0$. The inner product (6) coincides with the p-adic inner product $(\cdot,\cdot)_\lambda$ and the states (7) with the rational coefficients is the (dense) common domain of Dirac's and our considerations. If we realize the state (7) as the element of $\mathcal{H}_\lambda$ and have the normalization condition $|f| = 1$, then we get the standard statistical interpretation for this state with the only remark that relative frequencies stabilize with respect to the p-adic metric. We propose a rigorous mathematical meaning for Dirac's main formula (8) by setting the p-adic probability $\mathbf{P}_p$ instead of the real probability **P**. The relative frequencies can oscillate between 0 and 1 with respect to the real metric [see computer statistical experiments in Khrennikov (1994a)] and there is no way to give a mathematical description within the usual formalism. Our description is new from the mathematical point of view (new approach to the theory of probability), but sufficiently standard from the physical point of view because we use, as usual, the frequency approach to a notion of probability and have a bridge to the reality with the aid of rational relative frequencies. Of course, there is also a new physical step. It is the extension of the superposition principle on the basis of the p-adic Hilbert space to include new quantum states.

Thus, *Dirac's states (7) are physical states, but correspond to a new class of random sequences.*

APPENDIX. EXAMPLE OF *p*-ADIC PROBABILITY DISTRIBUTION WITH RATIONAL VALUES AND NEGATIVE PROBABILITIES OF SOME EVENTS

Let Ω be the standard Bernoulli probability space; it is the space of sequences $\omega = (\omega_j)$, $\omega_j = 0, 1$. Let $I = \cup I_n$, where I_n is the set of all vectors of length n with coordinates 0, 1. Let $i \in I_n$ and $B_i = \{\omega \in \Omega: \omega_1 = i_1, \ldots, \omega_n = i_n\}$; it is a cylindrical subset. Then the standard Bernoulli probability is defined by $\mu(B_i) = 1/2^n$. It is an additive set function and it can be extended to the standard Bernoulli probability μ_∞ on the σ-algebra generated by $\{B_i\}$.

But we are interested in another extension of μ. As μ assume its values in Q, it is also possible to consider it as a Q_p-valued measure. For example, it is possible to choose $p = 3$. It can be extended to a bounded Q_3-valued measure μ_3, 3-adic probability. We note that Ω is isomorphic to the unit ball $U_1(0) = \{x \in Q_2: |x|_2 \leq 1\}$ of Q_2 [see the 2-adic expansion (1)] $x \in U_1(0)$ iff there is no negative degree of 2 in (1) and $x \to \omega$, $\omega_1 = x_0, \ldots, \omega_n = x_{n+1}, \ldots$. According to this isomorphism, B_i is a ball

$$U_{2^{-n}}(a) = \{x \in Q_2: |x - a|_2 \leq 2^{-n}\}$$

where a is an arbitrary point of Q_2 with the property $a_0 = i_1, \ldots, a_{n-1} = i_n$.

Remark. We cannot discuss here the theory of p-adic-valued measures (see Schikhof, 1984). It is a result of this theory that μ can be extended to a bounded measure with values in every Q_p, $p \neq 2$.

It is interesting that the probabilities μ_∞ and μ_3 coincide for all events which depend of a finite number of experiments (B_i). Thus, we cannot distinguish these two distributions by experiment.

Now let us introduce on Ω a density ρ: $\Omega \to Q$ to generate a new probability distribution. Set $i_{n0} = (0, \ldots, 0) \in I_n$ and $S_n = B_{i_{n0}} \backslash B_{i_{(n+1)0}}$. Then

$$\Omega = \bigcup_{n=0}^{\infty} S_n, \qquad S_n \cap S_m = \emptyset, \qquad n \neq m \tag{A1}$$

Set $\rho(\lambda) = 3^n$ on S_n. Let us try to normalize a measure $d\nu(\lambda) = \rho(\lambda)\, d\mu(\lambda)$. First, in the real case, $\int_\Omega d\nu_\infty(\lambda) = \int_\Omega \rho(\lambda)\, d\mu_\infty(\lambda) = \infty$. Thus, it is impossible to normalize this distribution and there is no standard probability distribution.

Now let us consider the 3-adic case. We note that $S_n = \{x \in Q_2: |x|_2 = 2^{-n}\}$ is a 2-adic sphere of radius 2^{-n} with center at zero. The representation (A1) is nothing else than the representation of the unit 2-adic ball as the union of spheres. We get

$$\delta = \int_\Omega d\nu_3(\lambda) = \sum_{n=0}^{\infty} 3^n(1/2^n - 1/2^{n+1})$$

This series converges in Q_3 as $|(3/2)^n|_3 = 1/3^n \to 0$ and in the p-adic case the series $\sum a_n$ converges **iff** $|a_n|_p \to 0$. We get $\delta = -1$. Thus, we can normalize the density $\rho(\lambda) \to \rho_{\text{norm}}(\lambda) = \rho(\lambda)/\delta$ and introduce the 3-adic-valued distribution $d\mathbf{P}_3(\lambda) = \rho_{\text{norm}}(\lambda)\, d\mu_3(\lambda)$. The 3-adic probabilities of events S_n are negative: $\mathbf{P}_3(S_n) = -3^n/2^{n+1}$, but the probabilities of some events are positive: for $B_{i_{n0}} = \cup_{k=n}^{\infty} S_k$ we get $\mathbf{P}_3(B_{i_{n0}}) = \sum_{k=n}^{\infty} \mathbf{P}_3(S_k) = (3/2)^n$.

How would it be possible to express the probability distribution $d\mathbf{P}_3(\lambda)$ in an intuitive way? Let us realize Ω as the segment $[0, 1]$, i: $\Omega \to R$, $\omega \to s = \sum_{j=1}^{\infty} \omega_j 2^{-j}$. Then $i(S_0) = [1/2, 1]$, $i(S_1) = [1/4, 1/2]$, $\ldots$, $i(S_n) = [1/2^{n+1}, 1/2^n]$, ... [$i$ is not an isomorphism; for example, $i((1, 0, \ldots, 0, \ldots)) = i((0, 1, \ldots, 1, \ldots)) = 1/2$]. The probability density $\rho(\lambda)$ on $[0, 1]$ has the form $\rho(\lambda) \equiv 3^n$ on the segment $[1/2^{n+1}, 1/2^n]$. Thus, this probability distribution is very quickly concentrated in the neighborhood of zero, so quickly that real numbers cannot describe this increase.

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